

Slicing the Fock space for state production and protection

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In this letter we present a protocol to engineer interactions confined to subspaces of the Fock space in trapped ions: we show how to engineer upper-, lower-bounded and sliced Jaynes-Cummings (JC) and anti-Jaynes-Cummings (AJC) Hamiltonians. The upper-bounded (lower-bounded) interaction acting upon Fock subspaces ranging from $|0\rangle$ to $|M\rangle$ ($|N\rangle$ to ∞), and the sliced one confined to Fock subspace ranging from $|M\rangle$ to $|N\rangle$, whatever $M < N$. Whereas the upper-bounded JC or AJC interactions is shown to drive any initial state to a steady Fock state $|N\rangle$, the sliced one is shown to produce steady superpositions of Fock states confined to the sliced subspace $\{|N\rangle, |N+1\rangle\}$.

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With the advent of quantum computation and communication, it has become mandatory the development of techniques for the strict control of the coherent manipulation of quantum states. Since the mid-1990s, much has been accomplished with techniques for engineering effective Hamiltonian [1] for preparation of non-classical states [2] and manipulation of their evolutions. Concurrently, we have witnessed notorious progress on techniques for controlling decoherence, with proposals like error correction [3], decoherence-free subspaces [4], reservoir engineering [5–8] and quantum feedback [10], all already implemented experimentally [11, 12].

More specifically, the search for steady Fock states has long been sought in the framework of quantum computation and communication [12], and has led recently to a noticeable result in cavity QED [12]: the generation of Fock states with photon number n up to 7 was driven, with probability around 0.4, by using a quantum feedback procedure to correct decoherence-induced quantum jumps. Nonequilibrium number states up to 2 photons were long before

prepared in cavity QED [13] as well as in most suitable platforms as ion traps [14] and, lately, in circuit QED [15] where number states up to 6 was achieved.

In the present letter, we bring to ion trap systems some of the above-mentioned elements to produce high-fidelity quasi-steady motional Fock states and steady superpositions of Fock states. The protocol proposed relies on two key ingredients: the engineering of interactions lying in specific subspaces of the Fock space and engineered reservoirs [6–8]. Specifically to the former, it is demonstrated how one can derive effective Jaynes-Cummings (*JC*) and anti-Jaynes-Cummings (*AJC*) interactions which confine a state evolution to the subspaces $|0\rangle$ to $|M\rangle$ or $|N\rangle$ to $|\infty\rangle$. Therefore, the spectral decomposition of the time evolution of such states will be upper and lower-bounded (*ub* and *lb*) in the Fock space. In addition, we also show a scheme for a slicing of the Fock space, tailoring the system interactions in order to have them confined to Fock subspaces ranging from $|M\rangle$ to $|N\rangle$, with $M < N$.

The engineering reservoir technique is required to produce a Lindblad absorption band, due to the *ub* *AJC* interaction, whose competition with the natural Lindblad emission terms can be adjusted to favor the absorptive process, thus leading any initial motional state to a quasi-steady Fock state $|N\rangle$. Moreover, when the sliced interaction is used to generate an equally sliced Lindblad superoperator acting upon the subspace $\{|N\rangle, |N+1\rangle\}$, we show that the parameters can be conveniently adjusted to drive any initial state to a given superposition $|\psi\rangle = c_N |N\rangle + c_{N+1} |N+1\rangle$, thus extending the scheme in Ref. [7] which applies to the specific subspace $\{|0\rangle, |1\rangle\}$.

We start with the Hamiltonian for the coupling between the electronic and motional degrees of freedom of the trapped ion, the former described by the raising $\sigma_+ = |e\rangle\langle g|$ and lowering $\sigma_- = \sigma_+^\dagger$ operators, $|g\rangle$ and $|e\rangle$ standing for the ground and excited states, respectively, and the latter described by the annihilation a and creation a^\dagger operators. In the resolved sideband regime —where the detuning between the electronic transition frequency ω_0 and the laser beam ω (used to couple the ionic degrees of freedom) is an integer k multiple of the vibrational frequency ν , i.e. $k = (\omega_0 - \omega)/\nu$ —, we obtain

$$H = \Omega e^{i\phi - \eta^2/2} \sigma_+ \sum_{l=0}^{\infty} \frac{(i\eta)^{2l+k}}{l!(l+k)!} (a^\dagger)^l a^l a^k + H.c.,$$

Ω being the Rabi frequency, η the Lamb-Dicke parameter and ϕ the phase of the laser field used to couple both ionic degrees of freedom. By tuning the laser beam to the first red (blue) sideband and working to second order in the Lamb-Dicke parameter, we derive the

interaction

$$H_{k=\pm 1} = \chi(\eta) [\mathbf{A}(\eta)\sigma_{\pm} + \mathbf{A}^{\dagger}(\eta)\sigma_{\mp}] ,$$

where $\chi(\eta) = \eta(1 - \eta^2/2)\Omega$ and $\mathbf{A}(\eta) = [1 - \eta^2 a^{\dagger}a/2]a$. Expanding the operators A and A^{\dagger} in the Fock space basis and adjusting the Lamb-Dicke parameter to $\eta^2 = 2/N$ [$\eta^2 = 2/(N-1)$] such that $\mathbf{A}^{\dagger}(\eta)|N\rangle = 0$ [$\mathbf{A}(\eta)|N\rangle = 0$], we readily note that $H_{k=\pm 1}$ is decomposed in a sum of upper- (*ub*) and lower-bound (*lb*) Hamiltonians, in the form

$$H_{k=\pm 1} = H_{\pm}^{(ub)} + H_{\pm}^{(lb)},$$

$$H_{\pm}^{(ub)} = \sum_{n=0}^{N-1} \chi_n (|n\rangle \langle n+1| \sigma_{\pm} + H.c.), \quad (1a)$$

$$H_{\pm}^{(lb)} = \sum_{n=N+1}^{\infty} \chi_n (|n\rangle \langle n+1| \sigma_{\pm} + H.c.), \quad (1b)$$

where $\chi_n = \sqrt{n+1}(1 - \eta^2 n/2)\chi(\eta)$ and the (*ub* or *lb*) Hamiltonians H_+ and H_- clearly stand for the JC and AJC interactions, respectively. These interactions become incommunicable when we prepared the vibrational state confined to the *ub* or the *lb* subspace, $|0\rangle$ to $|N\rangle$ or $|N+1\rangle$ to $|\infty\rangle$: The evolution of any prepared state $\rho = \sum_{m,n=0}^N p_{mn} |m\rangle \langle n|$ or $\rho = \sum_{m,n=N}^{\infty} p_{mn} |m\rangle \langle n|$, whatever the electronic state is, remains indeed confined within the *ub* or the *lb* subspace. The considered second-order approximation holds for N up to 10 with typical parameters of ion trap experiments, allowing our technique to be applied for several states of Fock space.

We have thus engineered JC or AJC interactions restricted to subspaces $|0\rangle$ to $|N\rangle$ or $|N+1\rangle$ to $|\infty\rangle$, whatever the integer N is, adjusted through the choice of η . Additionally to that feature, one can envision a use of the engineered interactions Eq. (1) in order to decouple the vibrational and internal atomic degrees of freedom. Indeed, by considering a bicromatic field (generated through the above considered laser plus an electro optical modulator), tuned to the first red and blue sidebands simultaneously, one obtains $H = H^{(ub)} + H^{(lb)}$, where

$$H^{(ub)} = \sum_{n=0}^{N-1} \chi_n (|n\rangle \langle n+1| + H.c.) \sigma_x, \quad (2a)$$

$$H^{(lb)} = \sum_{n=N+1}^{\infty} \chi_n (|n\rangle \langle n+1| + H.c.) \sigma_x. \quad (2b)$$

Observe that a choice for the atomic state as an eigenstate of σ_x effectively decouples the vibrational and internal degrees of freedom, enabling one to directly select the *ub* or *lb* vibrational subspaces.

Now we turn our attention to engineer a Hamiltonian which confines the dynamics of the vibrational state to a slice of the Fock space. For that, let us consider again two laser beams. One of them electro-optically tuned to the carrier (Ω_1) as well as the first red (Ω_3) and blue (Ω_4) sidebands, while the other (Ω_2) is tuned to resonance with the electronic transition. Working again to second order in the Lamb-Dicke parameters adjusted such that $\eta_1^2 = \eta_4^2 = 2/(N+1)$, $\eta_2^2 = 2/N$, and $\eta_3^2 = 2/(N-1)$, it follows the interaction

$$H = \Omega (\mathbf{B}\sigma_+ + \mathbf{B}^\dagger\sigma_-), \quad (3)$$

where $\mathbf{B} = \bar{\Omega}_1\mathbf{N}_1 + \bar{\Omega}_2\mathbf{N}_2 + \bar{\Omega}_3\mathbf{N}_3a + \bar{\Omega}_4a^\dagger\mathbf{N}_4$ and $\mathbf{N}_j = \mathbf{1} - \eta_j^2 a^\dagger a/2$. We have also adjusted the Rabi frequencies to obtain $\bar{\Omega}_1 = \Omega_1/\Omega = (N+1)\sqrt{N+1}/(N-1)$, $\bar{\Omega}_2 = \Omega_2/\Omega = N\sqrt{N+1}/(N+1)$, and $\bar{\Omega}_3 = \Omega_3/\Omega$, $\bar{\Omega}_4 = \bar{\Omega}_3^{-1}$. It is straightforward to verify that, for a prepared state $|\psi\rangle = c_N|N\rangle + c_{N+1}|N+1\rangle$ with $c_N/c_{N+1} = \bar{\Omega}_3$, the evolution governed by Hamiltonian (3) confines $|\psi\rangle$ to the subspace $\{|N\rangle, |N+1\rangle\}$. Although this Hamiltonian does not apply for $N=0$ or 1 because of our choice of the Lamb-Dicke parameters, the case $N=1$ can be implemented by considering engineering interactions (confined to the subspace $\{|1\rangle, |2\rangle\}$) using two laser beams, each electro-optically tuned to two carrier transitions and the first blue sideband. The first laser is set to be within the Lamb-Dicke regime, with the phase adjusted to introduce a global phase factor $e^{i\pi}$ in all transitions, while the second one has to be treated under a second order approximation in appropriately adjusted Lamb-Dicke parameters. We finally stress that, under the same considerations used to derive the interaction (3), we may engineer a Hamiltonian to confine the evolution of any initial state $\rho = \sum_{m,n=N}^{N+\ell} p_{mn} |m\rangle\langle n|$ to the subspace $\{|N\rangle, \dots, |N+\ell\rangle\}$ using ℓ additional laser beams. For this purpose we set the coefficients of the superposition $|\psi\rangle$ so that $\mathbf{B}|\psi\rangle = \lambda|\psi\rangle$, and as we will check below, with the additional condition $\lambda = 0$ the confined state is as well protected against decoherence.

Turning to the applications of the above derived (seven) Hamiltonians, we first present a method to protect a Fock state which relies on engineered reservoir, where an auxiliary decaying system (here the electronic levels of the ion) is used to protect the state of the system of interest (the ionic vibrational degrees of freedom). When considering the *ub AJC* interaction $H_-^{(ub)}$, it is straightforward to obtain, by analogy with Refs. [6–8], the engineered master equation

$$\mathcal{L}_{eng}\rho = \frac{\Gamma}{2} (2\mathbf{A}^\dagger\rho\mathbf{A} - \mathbf{A}\mathbf{A}^\dagger\rho - \rho\mathbf{A}\mathbf{A}^\dagger), \quad (4)$$

with the effective damping rate $\Gamma = 4\chi^2/\kappa$, κ being the damping rate of the internal excited state of the ion. As for interaction Eq. (1b), the action of this superoperator, derived by getting rid of the electronic degrees of freedom, is confined to the *ub* vibrational subspace. Analyzing the complete equation $\dot{\rho} = \mathcal{L}_{eng}\rho + \mathcal{L}\rho$, with $\mathcal{L}\rho = [(1 + \bar{n})\gamma/2] (2a\rho a^\dagger - \rho a^\dagger a - a^\dagger a \rho) + (\bar{n}\gamma/2) (2a^\dagger \rho a - \rho a a^\dagger - a a^\dagger \rho)$ standing for the contribution of the thermal environment, it is not difficult to conclude that under the condition $\Gamma \gg \gamma$, any initial state $\rho = \sum_{m,n=0}^N p_{mn} |m\rangle \langle n|$ is asymptotically driven to a quasi-steady Fock state $|N = 2/\eta^2\rangle$. This occurs because the engineered contribution $\mathcal{L}_{eng}\rho$, confined to the subspace $|0\rangle$ to $|N\rangle$, prevails over the action of the thermal environment. In Fig. 1, starting with the vibrational thermal state $\rho_{th} = \sum_n \bar{n}^n / (1 + \bar{n})^{1+n} |n\rangle \langle n|$ ($\bar{n} \approx 0.01$ being the typical average occupation number and k_B being the Boltzmann constant) and adjusting $\eta^2 = 2/M$, we present the evolutions of the fidelity $\mathcal{F}(t) = \text{Tr} |M\rangle \langle M| \rho(t)$ against γt , considering typical vibrational decay rate $\gamma_0 \sim 10$ Hz (where $\gamma = \gamma_0(1 + M)^{0.7}$ [9]), $\kappa \sim 4 \times 10^6$ Hz, and $\Gamma \sim 10^4 \gamma$. As shown by the black and grey dotted lines, the vibrational mode has been driven to steady Fock states $M = 5$ ($\eta^2 = 0.4$, $\Omega = 1.2 \times 10^6$ Hz) and 10 ($\eta^2 = 0.2$, $\Omega = 1.8 \times 10^6$ Hz), with significantly high fidelities, around 0.98, up to the relaxation time. In the inset of Fig. 1 we plot the Mandel Q -factor to inform us how close are the achieved steady states of the desired Fock states $|5\rangle$ and $|10\rangle$, for which $Q = -1$. We verify that the steady state reached with $M = 5$ is significantly closer to a Fock state, showing $Q = -0.88$. However, the state reached with $M = 10$ shows $Q = -0.77$, a value that starts to deviate significantly from the desired $Q = -1$ even though this state exhibits a fidelity around that reached with $M = 5$ and presents an unequivocally sub-poissonian statistics.

We observe that, although the interaction (1a) is not suited for state protection (since the thermal reservoir inevitably drives to the vacuum any vibrational state initially confined to the *lb* subspace), it is perfectly suited, as well as all other interactions here engineered, for the implementation of quantum-scissors device for optical state truncation [19]. Using again the protocol for engineering reservoir [6–8], we start from the Hamiltonian (3) to obtain the engineered master equation

$$\mathcal{L}_{eng}\rho = \frac{\tilde{\Gamma}}{2} (2\mathbf{B}\rho\mathbf{B}^\dagger - \mathbf{B}^\dagger\mathbf{B}\rho - \rho\mathbf{B}^\dagger\mathbf{B}), \quad (5)$$

with $\tilde{\Gamma} = 4\Omega^2/\kappa$. Similarly to what happens with Eq. (4), the action of the Liouvillian (5) is confined to the sliced subspace $\{|N\rangle, |N+1\rangle\}$, with $N \neq 0, 1$, defined by the Hamiltonian

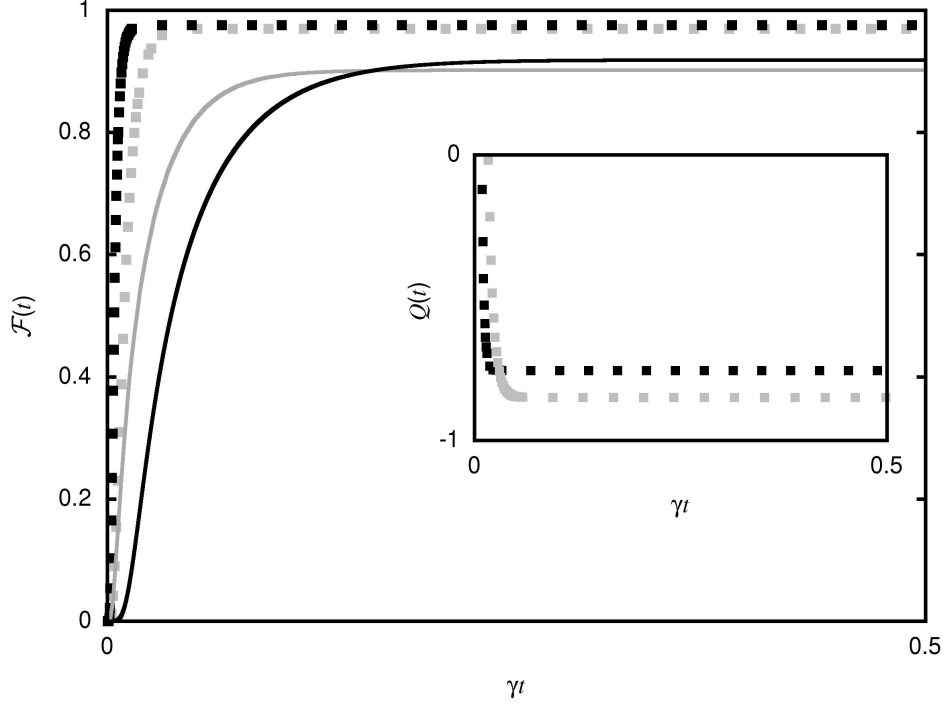


FIG. 1: Considering typical $\gamma_0 \sim 10$ Hz, $\kappa \sim 4 \times 10^6$ Hz, and $\Gamma \sim 10^4 \gamma$, the black and grey dotted lines indicate the evolutions of the fidelity $\mathcal{F}(t) = \text{Tr} |M\rangle \langle M| \rho(t)$ against γt for the Fock states $M = 5$ ($\eta^2 = 0.4$, $\Omega = 1.2 \times 10^6$ Hz) and 10 ($\eta^2 = 0.2$, $\Omega = 1.8 \times 10^6$ Hz), respectively, when starting from the thermal state $\bar{n} \approx 0.01$. Starting from the same thermal state and $\tilde{\Gamma} \sim 10^4 \gamma$, the black and grey solid lines indicate the evolutions of the fidelity $\mathcal{F}(t) = \text{Tr} |\psi\rangle \langle \psi| \rho(t)$ against γt , for the cases $M = 4$ and 9 , for which we have adjusted $\Omega \sim 5.9 \times 10^5$ Hz and $\Omega \sim 7.3 \times 10^5$ Hz, respectively. In the inset we display the evolutions of the Mandel's Q factor for the generation of the Fock states $M = 5$ and 10

from which it was engineered. Turning now to the equation $\dot{\rho} = \mathfrak{L}_{eng}\rho + \mathcal{L}\rho$, we verify that under the condition $\tilde{\Gamma} \gg \gamma$ —causing the contribution $\mathfrak{L}_{eng}\rho$ to prevail over $\mathcal{L}\rho$ —any superposition $|\psi\rangle = c_M|M\rangle + c_{M+1}|M+1\rangle$, where $\mathbf{B}|\psi\rangle = 0$, is protected against decoherence. Starting again with the typical values $\bar{n} \approx 0.01$ and $\tilde{\Gamma} \sim 10^4\gamma$, and adjusting $\eta_1^2 = \eta_4^2 = 2/(M+1)$, $\eta_2^2 = 2/M$, $\eta_3^2 = 2/(M-1)$, we also present in Fig. 1 the evolutions of the fidelity $\mathcal{F}(t) = \text{Tr}|\psi\rangle\langle\psi|\rho(t)$ against γt , for the cases $M = 4$ and 9 , for which we have adjusted $\Omega \sim 5.9 \times 10^5\text{Hz}$ and $\Omega \sim 7.3 \times 10^5\text{Hz}$, respectively. We verify that the vibrational mode has been driven to the equilibrium superposition $|\psi\rangle$, with exceptional high fidelity, around 0.90 , as shown by the black and grey solid lines. We finally observe that while the protection of superposition states is based on the protocol originally proposed in Ref. [6] and adopted in Ref. [7], our protocol for the protection of Fock states clearly differs from that in Ref. [6]. In fact, the protection of a superposition state $|\psi\rangle$ demands the eigenvalue equation $\mathbf{B}|\psi\rangle = 0$ as required in Ref. [6]. However, although the condition $\mathbf{A}^\dagger(\eta)|N\rangle = 0$ is automatically fulfilled to generate our required *ub* Hamiltonian, it is only a necessary condition. Our protocol for the protection of Fock states also demands the dynamics to be confined within the *ub* subspace during the whole time evolution.

We have thus presented an original protocol to slice the Fock space, i.e., to engineer upper-, lower-bounded and sliced JC and AJC Hamiltonians, which are confined to subspaces of the Fock space. These Hamiltonians are used to produce quasi-steady Fock states $|N\rangle$ and steady superpositions of Fock states confined to the sliced subspaces $\{|N\rangle, |N+1\rangle\}$. Our protocol can also be used for the implementation of quantum scissors, which shows its suitability in the implementation of quantum logical devices and to test the foundations of quantum mechanics.

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